A UNIQUENESS THEOREM FOR A CLASS OF INVERSE THERMAL-CONDUCTION PROBLEMS INVOLVING TEMPERATURE MEASUREMENTS AT INTERNAL POINTS

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A uniqueness theorem is formulated for a coefficient inverse problem in thermal conduction.

In the numerical solution of coefficient-type inverse thermal-conduction problems ITP [1], major importance attaches to uniqueness theorems, since they guarantee that experiment will be informative. The most readily available additional information required to solve the ITP lies in measuring the temperatures at certain internal points. The corresponding ITP may be called internal (in Lavrent'ev's terminology). A uniqueness theorem has been proved in [2] for an internal ITP in the derivation of one coefficient. In practice, all the coefficients are usually unknown. In [3], uniqueness was proved for an internal ITP for the simultaneous determination of two coefficients on the assumption that they are piecewise analytic and subject to the condition that there is a standard specimen with known properties within the body.

We introduce the following symbols: $n \geqslant 1$ an integer, $q(z) = (q_1(z), \ldots, q_n(z))$ a vector function continuous for $z \in R$, where R is a numerical straight line. For integer $m > 0$ R^m is an m-dimensional euclidean space. Let $b,\ \tau_m = \text{const} > 0$, $Q_{\tau_m} = (0,\ b) \times (0,\ \tau_m)$, $\overline{Q}_{\tau_m} = [0,\ b] \times [0,\ \tau_m]$, $C^{4,2}(\overline{Q}_{\tau_m})$ be the set of functions $u(x, \tau)$, having derivatives continuous in $\overline{Q}_{\tau_m}$:

$$\frac{\partial^{s+k}}{\partial x^s \partial \tau^k} u(x,\ \tau), \quad s + 2k \leqslant 4.$$

We consider the following ITP.

Problem. Let $F \in C^4(R^{n+5})$,

$$\frac{\partial F}{\partial z}(z,\ y) \neq 0, \quad \forall z \in R, \quad \forall y \in R^{n+4}.$$

Let the function $T \in C^{4,2}(\overline{Q}_{\tau_m})$ satisfy the following differential equation in Q_{τ_m}:

$$\frac{\partial T}{\partial \tau} = F\left(\frac{\partial^2 T}{\partial x^2},\ \frac{\partial T}{\partial x},\ T,\ x,\ \tau,\ q(T)\right) \tag{1}$$

and the conditions

$$T(x,\ 0) = f(x), \quad x \in (0,\ b), \tag{2}$$

$$T(0,\ \tau) = \mu(x), \quad \tau \in (0,\ \tau_m), \tag{3}$$

$$T(x_i,\ \tau) = \varphi_i(\tau), \quad \tau \in (0,\ \tau_m),\ i = 0,\ \ldots,\ n, \tag{4}$$

where $\{x_i\}_{i=0}^{n}$ is a specified set of points in the segment $(0, b)$, $0 < x_0 < \ldots < x_n < b$; we have to determine the vector function $(T, q(T))$.

An example of (1) is provided by the general heat-conduction equation

$$c(T)\frac{\partial T}{\partial \tau} = \frac{\partial}{\partial x}\left(\lambda(T)\frac{\partial T}{\partial x}\right) + k(T)\frac{\partial T}{\partial x} + H(T),$$

$$c \geqslant c_0, \quad \lambda \geqslant \lambda_0, \quad c_0,\ \lambda_0 = \text{const} > 0. \tag{5}$$

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In that case we have

$$F=\frac{1}{c(T)}\left[\frac{\partial}{\partial x}\left(\lambda(T)\frac{\partial T}{\partial x}\right)+k(T)\frac{\partial T}{\partial x}+H(T)\right].$$

We assume that $T(x, \tau)$ and $\varphi_i(\tau)$ increase wth x and τ correspondingly. More precisely:

$$\frac{\partial T}{\partial x}(x,\tau)\geqslant\beta \ \text{for}\ (x,\tau)\in\{x_0<x<b,\ 0<\tau<\tau_m\},$$

$$\frac{\partial T}{\partial x}(x,\tau)\geqslant 0 \ \text{for}\ (x,\tau)\in\{0<x<x_0,\ 0<\tau<\tau_m\},\tag{6}$$

$$\varphi_i'(\tau)\geqslant\beta,\quad i=0,\ \dots,\ n;\quad \beta=\text{const}>0.\tag{7}$$

From (6) and (7) we get specifications for the experiment, which should be done with heating. Conditions (6) are usually physically evident [4].

We introduce the symbols $\alpha_0=f(0)$, $\alpha_1=f(b)$, $\alpha_2=\varphi_1(\tau_m)$, $D_{\alpha_2}=\{(x,\tau)\in Q_{\tau_m}\mid T(x,\tau)<\alpha_2\}$, and let $s_i(z)$ be a function inverse to $\varphi_i(\tau)$ for $z\in[\varphi_i(0),\ \varphi_i(\tau_m)]$.

Theorem 1. Let $\alpha_1<\alpha_2$, and let $q(z)$ is known for $z\in(\alpha_0,\ \alpha_1)$ and for n = 1

$$\frac{\partial F}{\partial q}\left(\frac{\partial^2 T}{\partial x^2},\ \frac{\partial T}{\partial x},\ T,\ x,\ \tau,\ q(T)\right)\Big|_{x=x_1}\neq 0,\quad \forall\tau\in[0,\ \tau_m],$$

and for $n\geqslant 2$

$$\det\|a_{ij}(z)\|_{i,j=1}^n\neq 0,\quad \forall z\in[\alpha_1,\ \alpha_2],\tag{8}$$

$$a_{ij}(z)=\frac{\partial F}{\partial q_j}\left(\frac{\partial^2 T}{\partial x^2},\ \frac{\partial T}{\partial x},\ T,\ x,\ \tau,\ q(T)\right)\Big|_{(x,\tau)=(x_i,s_i(z))}.\tag{9}$$

Then we confine not more than one vector function $(T,\ q(T))\in C^{4,2}(\overline{D}_{\alpha_2})\underbrace{C[\alpha_0,\ \alpha_2]\times\dots\times C[\alpha_0,\ \alpha_2]}_{n}$, satisfying (1)-(4). The region D_{α_2} is determined uniquely.

Note. In Theorem 1, it is envisaged that T and q_i are uniquely determined on the sets D_{α_2} and $(\alpha_0,\ \alpha_2)$ correspondingly. In fact, we consider the following situation here: at low temperatures, i.e., for $\tau\in(-r,\ 0)$, $r=\text{const}>0$, the functions $q_i(T)$ can be taken approximately as constants (or as linear ones). Then one can readily prove the uniqueness theorem in that case, e.g., for (5). At high temperatures, i.e., for $\tau>0$, we can use Theorem 1. The proof of Theorem 1 is based on a substantially modified method taken from [5-7] (see also [8]).

Conditions (8) and (9) for (5) imply the following:

1) in the case of determining c and H, $\varphi_1'(s_1(z))\neq\varphi_2'(s_2(z))$, $z\in[\alpha_1,\ \alpha_2]$. This inequality is checked directly, since the functions $\varphi_1,\ \varphi_2,\ s_1,\ s_2$ are known;

2) in the case of determining λ and H,

$$\frac{\partial T}{\partial x}(x_1,\ s_1(z))\neq\frac{\partial T}{\partial x}(x_2,\ s_2(z)),\quad z\in[\alpha_1,\ \alpha_2];\tag{10}$$

3) in the case of determining all the coefficients

$$\begin{vmatrix}\varphi_1'(s_1(z)) & \left[\frac{\partial T}{\partial x}(x_1,\ s_1(z))\right]^2 & \frac{\partial T}{\partial x}(x_1,\ s_1(z)) & 1\\ \varphi_2'(s_2(z)) & \left[\frac{\partial T}{\partial x}(x_2,\ s_2(z))\right]^2 & \frac{\partial T}{\partial x}(x_2,\ s_2(z)) & 1\end{vmatrix}\neq 0,\tag{11}$$

$$\begin{vmatrix} \varphi_3'(s_3(z)) & \left[\frac{\partial T}{\partial x}(x_3,\ s_3(z))\right]^2 & \frac{\partial T}{\partial x}(x_3,\ s_3(z)) & 1 \\ \varphi_4'(s_4(z)) & \left[\frac{\partial T}{\partial x}(x_4,\ s_4(z))\right]^2 & \frac{\partial T}{\partial x}(x_4,\ s_4(z)) & 1 \end{vmatrix}$$

$$z \in [\alpha_1,\ \alpha_2].$$

Conditions (10) and (11) can be checked on numerical computation. The determinant of (8) is considered along the isolines $\{(x, \tau) \mid T(x, \tau) = z\}$.

NOTATION

$b,\ \tau_m = \text{const} > 0,\ Q_{\tau_m} = (0,\ b)\times(0,\ \tau_m);\ C^{4,2}(\overline{Q}_{\tau_m})$, set of functions u(x, τ) with continuous derivatives $\frac{\partial^{s+k}}{\partial x^s \partial \tau^k} u(x,\ \tau),\ s + 2k \leqslant 4;\ T(x,\ \tau)$ in $\overline{Q}_{\tau_m}$; T(x, τ), solution to the equation having the physical meaning of temperature; $q(z) = (q_1(z), \ldots, q_n(n))$, vector function consisting of unknown coefficients.

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